Julia Sets for Halley's Method, Double Standard Maps and Maps with Holes.

by

Diego Leonardo Lugo Ojeda

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Thesis supervisors : Dr. Gerardo Honorato (UV)
Dr. Rafael Alcaraz Barrera (UASLP - México)

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Preface

The present thesis is the result of two main works that are framed, roughly speaking, in two areas of Dynamical Systems: Complex Dynamics and One Dimensional Real Dynamics. On the first subject, we prove the non-connectedness of the Julia Set of Halley's Method iteration function for finding roots of complex functions of the form $f(z) = p(z)e^{q(z)}$; where p and q are non-constant polynomials. The second work establishes the first study of a family of one dimensional nonlinear maps within the theory of Maps with holes. The main result in this context is the obtention of an inferior bound for the Hausdorff dimension of a certain invariant set (known as *Survival set* which will be defined in the Chapter 4.1). Formally speaking, the two main results presented in this work are:

Theorem A. *There exist complex polynomials p, q such that the Julia set of Halley's method applied to a map of the form $F = pe^q$ is disconnected.*

Theorem B. *Let $f_{\alpha,\beta}$ be an expanding DSM with $\beta \neq 0$ and $(a, b) \subset \mathbb{S}^1$ a fixed hole. Then*

$$\overline{\gamma}\dim_{\mathrm{H}}(S_{f_{\alpha,\beta}}(a, b)) \leq \dim_{\mathrm{H}}(S_D(\phi(a), \phi(b))) \leq \frac{1}{\gamma}\dim_{\mathrm{H}}(S_{f_{\alpha,\beta}}(a, b)).$$

Here, $\gamma, \overline{\gamma} > 0$ are the Hölder constants for ϕ, ϕ^{-1} respectively.

In order to facilitate the reading of this Thesis, it is structured in two parts; each one can be read separatedly (only Theorem 2.1.5 is referenced in both parts). The structure is the following

- Part I: Complex Dynamics, which is divided in three Chapters:
  - Chapter 1: we set the basic definitions, notation and we state some relevant results for Dynamics of root-finding methods. Finally we present the statement of results.
  - Chapter 2: we present all the tools of Complex Dynamics that we will use along the next chapter.

  - Chapter 3: we state and prove our results regarding the Julia Set for the Halley's Method, including Theorem A.

  As a result of this work, we obtain an indexed publication in 2022 jointly with Patricio Cumsille, Gerardo Honorato and Juan Gonzalez-Marín in *Dynamical Systems: An International Journal.* —see [34].

- Part II: One Dimensional Dynamics, divided in three Chapters:

  - Chapter 4: we set the basic definitions, notation and some relevant results for One Dimensional Dynamics.
  - Chapter 5: we present all the tools of One Dimensional Dynamics the we will use along the next chapters
  - In Chapter 6: we state and prove our results regarding the family of Double Standard Maps when an open interval of the circle is considered as a hole.