Introduction

A real structure on a connected n-dimensional complex manifold X is provided by an anti-holomorphic involution; its fixed points being the corresponding real points and the set of real points being its real part. The real part is a natural invariant and the problem of computing is called the Harnack's problem. Two problems arise in this context:

- The computation of the number of connected components of the real part of a real structure, together the description of each of its components.
- The determination and description of the different real structures that X may have, up to conjugation by holomorphic automorphisms of X.

If X is a closed Riemann surface of genus g ($g \geq 0$), then the real part of a real structure on X has at most $g + 1$ connected components; this was proved by Harnack himself. Later, it was proven that

- For g even and greater than zero, there are at most four real structures up to holomorphic conjugacy, this was proved by Bujalance and Izquierdo (in genus zero, there are exactly two)
- For g odd, although there are not absolute bounds, there are upper ones in terms of g; this fact was shown by Bujalance, Gromadzki and Izquierdo.

It seems that there is no known similar results for X compact complex manifold of dimension $n \geq 2$, except in some particular situations; for instance, the n-dimensional complex projective space $\mathbb{P}^n_{\mathbb{C}}$, where $n \geq 2$ even, and the complex tori.

A non-compact example is given by the Teichmüller space $\mathcal{T}_g$, where $g \geq 1$, which is a finite dimensional simply-connected complex manifold parametrizing isomorphism classes of marked closed Riemann surfaces of genus g. The description and classification of its real structures (and their corresponding real parts) have been done by several authors; particular importance in this task is due to Seppäla . The moduli space $\mathcal{M}_g$, that parametrizes isomorphism classes of closed Riemann surfaces of genus $g \geq 2$, is a complex orbifold of complex dimension $3(g-1)$ and it carries a natural real structure; its real points are the classes of Riemann surfaces isomorphic to their conjugated ones. A description and classification of this real part was developed by Earle, Natanzon and Seppäla.

Other interesting examples of non-compact finite dimensional connected complex manifolds (and, in general, non-simply connected ones) are given by the quasiconformal deformation space $Q(K^+)$ of a finitely generated Kleinian group K^+ with non-empty region of discontinuity.

If K^+ is a torsion-free co-compact Fuchsian group acting on the unit disc $\mathbb{D}$, then $Q(K^+)$ is isomorphic to $\mathcal{T}_g \times \mathcal{T}_g$, where g is the genus of the closed Riemann surface $\mathbb{D}/K^+$. So, a description of its real structures can be obtained by the knowledge of those of $\mathcal{T}(S)$.

For more general kind of finitely generated Kleinian groups, the description and classification of its real structures up to holomorphic conjugation seems to be unknown. In this thesis, we restrict to the case when K^+ is a Schottky group of rank $g \geq 1$.

As any two Schottky groups of the same rank are quasiconformally conjugated, their quasiconformal deformation spaces are isomorphic. As a result of Bers's work, we know that a model of $Q(K^+)$, for K^+ a Schottky group of rank g, is given by the marked Schottky space $\mathcal{MS}_g$. The space $\mathcal{MS}_1$ is isomorphic to the punctured unit disc $\mathbb{D}^* = \{0 < |z| < 1\}$ and, for $g \geq 2$, $\mathcal{MS}_g$ is a connected and non-simply connected complex manifold of dimension $3(g-1)$.

The fact that $\mathcal{MS}_1$ is isomorphic to $\mathbb{D}^*$ asserts that each of its real structures is given by the reflection on a diameter. In particular,

- every real structure has real points,
- the real part of each real structure has two connected components (each one being an arc), and
- all real structures are conjugated in its group of holomorphic automorphisms (rotations about the origin) to the canonical real structure $J_1(z) = \overline{z}$

The main objective of this thesis is to provide a description of the real structures, up to conjugation by holomorphic automorphisms, and their real parts, of $\mathcal{MS}_g$ for $g \geq 2$.

We will first observe that each real point of a real structure of $\mathcal{MS}_g$ is provided by an extended Schottky group of rank g. Simple examples of extended Schottky groups are provided by those Kleinian groups generated by the complex conjugation and a Fuchsian Schottky group acting on the upper-half plane (called real Schottky groups); these kind of groups have been used by Bobenko to study certain partial differential equations and they also provide uniformizations of compact Klein surfaces with boundary.

A structural decomposition of extended Schottky groups, in terms of the Klein-Maskit combination theorems was obtained by Gromadzki and Hidalgo. This structural decomposition permits to observe that topologically conjugated extended Schottky groups are also quasiconformally conjugated, and also to obtain the number M_g of topologically non-conjugated extended Schottky groups of a fixed rank g.

We will see that each extended Schottky group of rank g induces in a natural way a real structure on $\mathcal{MS}_g$ with non-empty real part for which one of its connected components is a real analytic copy of its quasiconformal deformation space. We prove that the converse holds and that real points of real structures on $\mathcal{MS}_g$ are in correspondence with extended Schottky groups. We prove four things:

- How we can identify a real point of the marked Schottky space $\mathcal{MS}_g$; even more, how we can identify a entire connected component of the real part of a real structure.
- The conditions under two real structures are corresponding extended Schottky group representing a real point on such a component conjugate in the group of holomorphic automorphisms of $\mathcal{MS}_g$.
- We find the exact number of non-conjugate real structures on $\mathcal{MS}_2$.
- If $g \geq 3$ and T_g is the number of conjugacy classes of order two elements of $\mathrm{Out}(F_g)$, then we find thee number of non-conjugate real structures on $\mathcal{MS}_g$ in terms of T_g.

This thesis is organized as follows.

In the first chapter, we will review some relevant preliminaries as Riemann surfaces, fundamental groups, universal coverings, quasiconformal maps and Teichmüller spaces.

In the second chapter, we provide the definitions of Möbius and extended Möbius transformations, and the important, particular case when they are Kleinian and extended Kleinian groups; also we discuss the uniformization theorem that let us to describe Riemann surfaces through Kleinian groups, the Alhfors' Finiteness Theorem that says us how to describe the quotient surface Ω_K/K, when K is an extended Kleinian group and decomposition theorem, given by Maskit. In the final sections, we define when the cases K is Schottky group and extended Schottky group, state a structural description of extended Schottky groups and compute the number of topologically conjugacy classes of extended Schottky groups of a fixed rank.

In the third chapter, we provide the definitions of a real structure of a complex manifold and its associate real part. Our focus will be in three important cases: compact Riemann surfaces of genus g, Teichmüller spaces of closed Riemann surfaces; these two first cases are completed solved. Finallly, we going to find and study some real structures of quasiconformal deformation spaces of function groups, which are induced by extended Kleinian groups.

In the fourth chapter, we provide the definition of marked Schottky groups and the marked Schottky space $\mathcal{MS}_g$ (observing that this is a model for the quasiconformal deformation of a Schottky group of rank g). Also, it is stated Marden's description of the group of automorphisms of $\mathcal{MS}_g$, which asserts that, for $g \geq 3$, the group of its holomorphic automorphisms is naturally isomorphic to the group $\mathrm{Out}(F_g)$ of exterior automorphisms of the free group of rank g. Then we discuss its real structures, introducing first the "canonical" one. We compute the real structures for $\mathcal{MS}_2$ and we count the number of conjugacy classes of real structures for $\mathcal{MS}_g$.

In the final chapter, we state and prove the main theorem.